

Relaxation to equilibrium driven via indirect control in Markovian dynamics*

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We characterize to what extent it is possible to modify the stationary states of a quantum dynamical semigroup, that describes the irreversible evolution of a two-level system, by means of an auxiliary two-level system. We consider systems that can be initially entangled or uncorrelated. We find that the indirect control of the stationary states is possible, even if there are not initial correlations, under suitable conditions on the dynamical parameters characterizing the evolution of the joint system.

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INTRODUCTION

In the past decades quantum mechanical systems have attracted lot of attention for their peculiar properties, that indicate they are good candidates for the implementation of outperforming technologies in the fields of information and computation[1]. In this spirit, some amazing protocols have been recently developed, as for example the computational algorithm for the factorization of a large number [2], or the schemes for teleportation [3] and quantum cryptography [4]. Many concrete physical systems have been proposed for the practical implementation of these ideas, as optical devices, cold trapped atoms, nuclear spins in magnetic fields (NMR), or quantum dots in electromagnetic cavities.

In all these cases, the largest obstacle to the implementation of stable and efficient schemes is represented by the unavoidable interaction of microscopic systems with the surrounding environment. Because of this interaction, the system dynamics is subject to loss of coherence, irreversibility and dissipation, and the appealing properties of quantum systems are usually lost or compromised during the time evolution.

Usually, the environmental action is accounted for by describing the dynamics of the system S through a Markovian one-parameter family of maps $\{\gamma_t; t \geq 0\}$, satisfying the semigroup property $\gamma_{t+s} = \gamma_t \circ \gamma_s$, with $t, s \geq 0$, with

$$\rho_S(t) = \gamma_t[\rho_S(0)], \quad (1)$$

where the statistical operator (or density matrix) ρ_S is an Hermitian, positive, unit trace operator, acting on the Hilbert space associated to the system, and representing

its state. This representation of the dynamics is not the most general, but it is well justified in many cases, in particular when the coupling between S and the environment can be considered weak. The generator L of the dynamics can be obtained by writing (1) in differential form, $\dot{\rho}_S = L[\rho_S]$, and it has the standard structure

$$L[\rho_S] = -i[H_S, \rho_S] + \sum_{i,j} c_{ij} \left(F_i \rho_S F_j^\dagger - \frac{1}{2} \{F_j^\dagger F_i, \rho_S\} \right), \quad (2)$$

where $H_S = H_S^\dagger$ is the system Hamiltonian, and the set $\{F_i; i\}$ satisfies $\text{Tr} F_i = 0$, $\text{Tr}(F_i F_j^\dagger) = \delta_{ij}$. The Kosakowski matrix $C = [c_{ij}]$ must satisfy $C^\dagger = C \geq 0$ in order to guarantee the complete positivity of the evolution, and then the physical consistency of the formalism [5, 6]. It encodes the microscopical details of the interaction between system and environment. The first term in the right hand side of (2) represents the coherent part of the evolution, and the generator of the system dynamics has this form whenever the interaction with the surrounding environment can be neglected. The corresponding time evolution is given by a group of reversible, unitary transformations. The second term is responsible for irreversibility and dissipation, since it produces a contraction map on the set of states, and, in some cases, relaxation to stationary states.

It is of fundamental relevance to study methods to fight against decoherence. When the environmental noise exhibits some particular symmetry properties, this task can be realized by encoding the relevant information in suitable *Decoherence Free Subspaces or Subsystems* unaffected by decoherence (for a review, see [7]). An active approach consist in directly affecting the system dynamics, in order to preserve its relevant properties or induce arbitrary manipulations. This *controlled evolution* is realized through some functions, entering the dynamics, that can be manipulated via external actions (see for example [8, 9, 10, 11, 12, 13, 14] for a geometric approach to controllability).

Several approaches to the control of a quantum system have been proposed in the past years. In the *open loop*

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schemes the control functions are a priori fixed (that is, they are independent on the state of the system). Conversely, in the *closed loop* control schemes, the control functions are updated in real time by feeding back some information about the actual state of the system, usually gained via an indirect continuous measurement (*quantum feedback* [15, 16, 17]).

The control functions usually affect the Hamiltonian of the system H_S , since the environmental action is usually uncontrollable. This approach is called *coherent control*, as it affects the coherent part of the dynamics. Motivated by different experimental scenarios, another control scheme has been introduced, in which an auxiliary system is used to manipulate the target system through their mutual interaction. This *indirect control* scheme is of relevance whenever the system dynamics cannot be directly accessed [18, 19, 20]. It represents a complementary approach to controllability, with interesting features concerning the purification of mixed states [21], and when applied to the dynamics of open systems [22], since it makes use of the correlations between the two subsystems, that can be created by the environmental action (described for the first time in [23]).

One of the unwished consequences of the environmental action on the system dynamics is, in many cases, the collapse of the system into a -in many cases, unique- equilibrium state, with a consequent reduction of the reachable sets, and loss of control. In this work we address the following question: is it possible to modify the stationary states of a *target* system T , evolving under a quantum dynamical semigroup, by means of an open-loop indirect control? In other words, we introduce an auxiliary system, a quantum *probe* P , couple it to T and consider the evolution of the joint system $S = T + P$, and finally discard P by taking into account only the degrees of freedom of T . Assuming that S is still described by a quantum dynamical semigroup, we study the stationary states of the system T alone, affected by P through the correlations between the two systems. The impact of both initial correlations, and correlations created during the joint evolution, is taken into account.

The plan of this work is the following. In Section I we review some algebraic tool for the determination of the stationary states of a quantum dynamical semigroup. In Section II we specify the dynamical settings considered in this paper and we derive the relevant algebraic quantities, introduced in Section I. In Section III we describe all the possible scenarios for the stationary states, in terms of the dynamical parameters characterizing the semigroup. In Section IV we summarize our results, describe their physical significance and finally conclude.

I. STATIONARY STATES OF QUANTUM DYNAMICAL SEMIGROUPS

In general, the second contribution in the right hand side of (2) leads to the appearance of attractors in the

state space of S , and consequently relaxation to equilibrium of the states of the system, absent if there is not interaction with the environment. A stationary state for the dynamics, ρ_S^∞ , is determined by the condition on the generator $L[\rho_S^\infty] = 0$. This is a system of linear equations that can be solved using standard algebraic tools. In this spirit, quantum dynamical semigroups have been classified in terms of their relaxing properties [24]. In the *uniquely relaxing semigroups*, there is a unique stationary state, and every initial state eventually collapses to it. In the *relaxing semigroups*, although every trajectory collapses to a fixed state, this state is not unique but depends on the initial conditions. Finally, in the *non-relaxing semigroups*, oscillatory solutions survive. Even if this method is very general, the resulting algebraic equations are complicate, therefore we will rely on a different approach.

For the Markovian dynamics (2), necessary conditions for the existence of stationary states and for the convergence of $\rho_S(t)$ to them have been derived in terms of the operators $\{V_i; i\}$ appearing in the diagonal form of (2),

$$L[\rho_S] = -i[H_S, \rho_S] + \sum_i \left(V_i \rho_S V_i^\dagger - \frac{1}{2} \{V_i^\dagger V_i, \rho_S\} \right). \quad (3)$$

The following theorem summarizes these conditions [25], and it will be the basis of our analysis.

Theorem 1 *Given the quantum dynamical semigroup (3), assume that it admits a stationary state ρ_0 of maximal rank. Defining $\mathcal{M} = \{H_S, V_i, V_i^\dagger; i\}'$, the commutant of the Hamiltonian plus the dissipative generators, and I the identity operator, the following conditions hold true:*

1. *If $\mathcal{M} = \text{span}(I)$, then ρ_0 is the unique stationary state. Moreover, if $\{V_i; i\}$ is a self-adjoint set with $\{V_i; i\}' = \text{span}(I)$, then for every initial condition $\rho_S(0)$*

$$\lim_{t \rightarrow +\infty} \rho_S(t) = \rho_0.$$

2. *If $\mathcal{M} \neq \text{span}(I)$, then there exist a complete family $\{P_n; n\}$ of pairwise orthogonal projectors such that $\mathcal{Z} = \mathcal{M} \cap \mathcal{M}' = \{P_n; n\}''$. If $\{V_i; i\}' = \mathcal{M}$, two extreme cases together with their linear superpositions may occur. If $\mathcal{Z} = \mathcal{M}$, then for every initial condition $\rho_S(0)$*

$$\lim_{t \rightarrow +\infty} \rho_S(t) = \sum_n \text{Tr}(P_n \rho_S(0) P_n) \frac{P_n \rho_0 P_n}{\text{Tr}(P_n \rho_0 P_n)}.$$

If $\mathcal{Z} = \mathcal{M}'$, then for every $\rho_S(0)$

$$\lim_{t \rightarrow +\infty} \rho_S(t) = \sum_n P_n \rho_S(0) P_n.$$

Therefore, in order to characterize the stationary states of a quantum dynamical semigroup, it is necessary to find a maximal rank stationary state ρ_0 , and to evaluate the algebras \mathcal{M} and \mathcal{M}' .

II. DYNAMICAL SETTINGS AND RELEVANT ALGEBRAS

We assume that $S = T + P$ is a bipartite system, where T and P are two copies of the same two-level system, separately interacting with a common environment according to the Markovian dynamics (2). The operators F_i are given by $F_i = \sigma_i \otimes \mathbb{I}$ for $i = 1, 2, 3$ and $F_i = \mathbb{I} \otimes \sigma_{i-3}$ for $i = 4, 5, 6$, where \mathbb{I} is the 2-dimensional identity operator, and σ_i , $i = 1, 2, 3$ are the Pauli operators. We consider the standard representation of these operators in which σ_3 diagonal. The matrix C has the form

$$C = \begin{bmatrix} A & B \\ B^\dagger & A \end{bmatrix}, \quad (4)$$

where $A = A^\dagger$ is the Kossakowski matrix for the system T (or P) alone, and B represents the dissipative coupling between the two parties. The form (4) is not the most general joint Kossakowski matrix. More complicate expressions should be taken into account when three-body contributions are relevant (common interactions between the two subsystems and the surrounding), and when the dissipative couplings of T and P to the environment are different (for example, in a non-homogenous medium). We will limit our attention to models satisfying (4); moreover, for simplicity, we will further assume $B = B^\dagger$. This assumption highly simplifies the mathematical formalism.

Following Theorem 1, we need to write C in diagonal form in order to find the operators V_i appearing in (3). This is achieved by means of the unitary transformation

$$UCU^\dagger = \text{diag}(\lambda_i, i = 1, \dots, 6), \quad (5)$$

where λ_i are the eigenvalues of C , U is of the form

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} \tilde{U} & \tilde{U} \\ -\hat{U} & \hat{U} \end{bmatrix} \quad (6)$$

and \tilde{U} , \hat{U} are unitary transformations such that

$$\begin{aligned} \tilde{U}(A+B)\tilde{U}^\dagger &= \text{diag}(\lambda_i^+, i = 1, 2, 3), \\ \hat{U}(A-B)\hat{U}^\dagger &= \text{diag}(\lambda_i^-, i = 1, 2, 3). \end{aligned} \quad (7)$$

The eigenvalues of C are ordered as $\lambda_i = \lambda_i^+$ for $i = 1, 2, 3$ and $\lambda_i = \lambda_{i-3}^-$ for $i = 4, 5, 6$. Comparing the generator forms (2) and (3), and using the notation $U = [u_{ij}]$, we have

$$V_i = \sqrt{\lambda_i} \sum_{k=1}^6 u_{ik}^* F_k, \quad i = 1, \dots, 6. \quad (8)$$

Following (6), it is possible to write

$$\frac{1}{\sqrt{\lambda_i}} V_i = \begin{cases} \mathbb{I} \otimes \tilde{\sigma}_i + \tilde{\sigma}_i \otimes \mathbb{I}, & i = 1, 2, 3 \\ \mathbb{I} \otimes \hat{\sigma}_{i-3} - \hat{\sigma}_{i-3} \otimes \mathbb{I}, & i = 4, 5, 6 \end{cases} \quad (9)$$

where we have defined

$$\tilde{\sigma}_i = \sum_{k=1}^3 \tilde{u}_{ik}^* \sigma_k, \quad \hat{\sigma}_i = \sum_{k=1}^3 \hat{u}_{ik}^* \sigma_k, \quad (10)$$

and we used the notation $\tilde{U} = [\tilde{u}_{ij}]$, $\hat{U} = [\hat{u}_{ij}]$. The operators in (10) satisfy $\text{Tr} \tilde{\sigma}_i = \text{Tr} \hat{\sigma}_i = 0$ and $\text{Tr}(\tilde{\sigma}_i \tilde{\sigma}_j^\dagger) = \text{Tr}(\hat{\sigma}_i \hat{\sigma}_j^\dagger) = \delta_{ij}$. They are self-adjoint if and only if the unitary operators \tilde{U} and \hat{U} are orthogonal.

The commutant of Theorem 1 can be expressed as

$$\{H_S, V_i, V_i^\dagger; i | \lambda_i \neq 0\}' = \bigcap_{i | \lambda_i \neq 0} \{V_i, V_i^\dagger\}' \cap \{H_S\}', \quad (11)$$

where only non-vanishing eigenvalues λ_i have to be considered, otherwise the corresponding V_i do not appear in the generator (3). Moreover, for a given i ,

$$\{V_i, V_i^\dagger\}' = \{v | v \in \{V_i\}', v^\dagger \in \{V_i\}'\}, \quad (12)$$

therefore we can limit our attention to the sets $\{V_i\}'$. We find convenient to consider separately the two kinds of contributions defined in (9). To begin with, we consider a fixed index i such that $\lambda_i^+ \neq 0$, and assume that the corresponding $\tilde{\sigma}_i$ is non-singular. In this case it can be written as

$$\tilde{\sigma}_i = \tilde{\mu}_i R_i \sigma_3 R_i^{-1} \quad (13)$$

where

$$R_i = R_i^{-1} = \frac{1}{\tilde{\nu}_i} \begin{bmatrix} \tilde{u}_{i3}^* + \tilde{\mu}_i & \tilde{u}_{i1}^* - i \tilde{u}_{i2}^* \\ \tilde{u}_{i1}^* + i \tilde{u}_{i2}^* & -\tilde{u}_{i3}^* - \tilde{\mu}_i \end{bmatrix}, \quad (14)$$

and

$$\tilde{\mu}_i^2 = \sum_j (\tilde{u}_{ij}^*)^2, \quad \tilde{\nu}_i = \sqrt{2\tilde{\mu}_i (\tilde{u}_{i3}^* + \tilde{\mu}_i)}. \quad (15)$$

Since $\mathbb{I} \otimes \tilde{\sigma}_i + \tilde{\sigma}_i \otimes \mathbb{I} = \tilde{\mu}_i \mathcal{R}_i (\mathbb{I} \otimes \sigma_3 + \sigma_3 \otimes \mathbb{I}) \mathcal{R}_i$, with $\mathcal{R}_i = R_i \otimes R_i$, it follows that

$$\{\mathbb{I} \otimes \tilde{\sigma}_i + \tilde{\sigma}_i \otimes \mathbb{I}\}' = \mathcal{R}_i \{\mathbb{I} \otimes \sigma_3 + \sigma_3 \otimes \mathbb{I}\}' \mathcal{R}_i \quad (16)$$

and then, after the explicit computation,

$$\{V_i\}' = \text{span}(\mathbb{I} \otimes \mathbb{I}, \mathbb{I} \otimes \tilde{\sigma}_i, \tilde{\sigma}_i \otimes \mathbb{I}, \tilde{\sigma}_i \otimes \tilde{\sigma}_i, \Omega^+, \Delta_i^-), \quad (17)$$

having defined the additional operators

$$\begin{aligned} \Omega^+ &= \sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2 + \sigma_3 \otimes \sigma_3, \\ \Delta_i^- &= \mathcal{R}_i (\sigma_1 \otimes \sigma_2 - \sigma_2 \otimes \sigma_1) \mathcal{R}_i. \end{aligned} \quad (18)$$

Notice that, in general, the operators in the right hand side of (17) are not self-adjoint, nor orthogonal each other in the Hilbert-Schmidt metric, since the transformation \mathcal{R}_i is not unitary. However, if the coefficients \tilde{u}_{ij}^* , $j = 1, 2, 3$, are real, $\tilde{\sigma}_i$ is self-adjoint and \mathcal{R}_i unitary (and self-adjoint). Consequently, in this case the basis of $\{V_i\}'$ is made of Hermitian, orthogonal operators.

Cases	Conditions	\mathcal{M} basis	\mathcal{M}' basis	$\mathcal{Z} = \mathcal{M} \cap \mathcal{M}'$	$\{P_n; n\}$
I	$n_+ = 1, A = A^T, B = A$	$\mathbb{I} \otimes \mathbb{I}, \mathbb{I} \otimes \tilde{\sigma}_i, \tilde{\sigma}_i \otimes \mathbb{I},$ $\tilde{\sigma}_i \otimes \tilde{\sigma}_i, \Omega^+, \Delta_i^-$	$\mathbb{I} \otimes \mathbb{I}, \tilde{\sigma}_i \otimes \tilde{\sigma}_i,$ $\mathbb{I} \otimes \tilde{\sigma}_i + \tilde{\sigma}_i \otimes \mathbb{I}$	\mathcal{M}'	$P_1 = \mathcal{R}_i \Pi_1 \mathcal{R}_i, P_2 = \mathcal{R}_i \Pi_4 \mathcal{R}_i,$ $P_3 = \mathcal{R}_i (\Pi_2 + \Pi_3) \mathcal{R}_i$
II	$n_+ = 1, A \neq A^T, B = A$	$\mathbb{I} \otimes \mathbb{I}, \Omega^+$	$\mathcal{M}' \supseteq \mathcal{M}$	\mathcal{M}	$P_1 = \Pi_1 + \Pi_4 + \Pi_+,$ $P_2 = \Pi_-$
III	$n_+ = n_- = 1, A = A^T,$ $B = \alpha A, \alpha \in \mathbb{R} \setminus \{-1, 1\}$	$\mathbb{I} \otimes \mathbb{I}, \tilde{\sigma}_i \otimes \tilde{\sigma}_i,$ $\mathbb{I} \otimes \tilde{\sigma}_i, \tilde{\sigma}_i \otimes \mathbb{I}$	$\mathcal{M}' \supseteq \mathcal{M}$	\mathcal{M}	$P_1 = \mathcal{R}_i \Pi_1 \mathcal{R}_i, P_2 = \mathcal{R}_i \Pi_2 \mathcal{R}_i,$ $P_3 = \mathcal{R}_i \Pi_3 \mathcal{R}_i, P_4 = \mathcal{R}_i \Pi_4 \mathcal{R}_i$
IV	$n_+ > 1, B = A$	$\mathbb{I} \otimes \mathbb{I}, \Omega^+$	$\mathcal{M}' \supseteq \mathcal{M}$	\mathcal{M}	$P_1 = \Pi_1 + \Pi_4 + \Pi_+,$ $P_2 = \Pi_-$

TABLE I: Relevant algebras for the determination of the stationary states for the two qubits system, under the assumption $H_S = 0$, in the non-trivial cases.

The commutants $\{V_i\}'$ are completely characterized for $i = 1, 2, 3$. Finally, $\{V_i, V_i^\dagger\}'$ can be found by considering (12):

$$\{V_i, V_i^\dagger\}' = \begin{cases} \{V_i\}', & \text{iff } \tilde{\sigma}_i = \tilde{\sigma}_i^\dagger; \\ \text{span}(\mathbb{I} \otimes \mathbb{I}, \Omega^+), & \text{otherwise.} \end{cases} \quad (19)$$

The corresponding sets for $i = 4, 5, 6$ can be found by applying the same procedure to $\hat{\sigma}_i$, assuming that $\lambda_i^- \neq 0$. The result is

$$\{V_i\}' = \text{span}(\mathbb{I} \otimes \mathbb{I}, \mathbb{I} \otimes \hat{\sigma}_i, \hat{\sigma}_i \otimes \mathbb{I}, \hat{\sigma}_i \otimes \hat{\sigma}_i, \Omega_i^-, \Delta_i^+), \quad (20)$$

where

$$\begin{aligned} \Omega_i^- &= \mathcal{S}_i(\sigma_1 \otimes \sigma_1 - \sigma_2 \otimes \sigma_2)\mathcal{S}_i, \\ \Delta_i^+ &= \mathcal{S}_i(\sigma_1 \otimes \sigma_2 + \sigma_2 \otimes \sigma_1)\mathcal{S}_i, \end{aligned} \quad (21)$$

and $\mathcal{S}_i = S_i \otimes S_i$, with

$$\hat{\sigma}_i = \hat{\mu}_i S_i \sigma_3 S_i^{-1}, \quad (22)$$

$$S_i = S_i^{-1} = \frac{1}{\hat{\nu}_i} \begin{bmatrix} \hat{u}_{i3}^* + \hat{\mu}_i & \hat{u}_{i1}^* - i \hat{u}_{i2}^* \\ \hat{u}_{i1}^* + i \hat{u}_{i2}^* & -\hat{u}_{i3}^* - \hat{\mu}_i \end{bmatrix}, \quad (23)$$

and

$$\hat{\mu}_i^2 = \sum_j (\hat{u}_{ij}^*)^2, \quad \hat{\nu}_i = \sqrt{2\hat{\mu}_i (\hat{u}_{i3}^* + \hat{\mu}_i)}. \quad (24)$$

Finally, in this case

$$\{V_i, V_i^\dagger\}' = \begin{cases} \{V_i\}', & \text{iff } \hat{\sigma}_i = \hat{\sigma}_i^\dagger; \\ \text{span}(\mathbb{I} \otimes \mathbb{I}), & \text{otherwise.} \end{cases} \quad (25)$$

If $\tilde{\sigma}_i$ (or $\hat{\sigma}_i$) is singular, the previous computations are not longer valid. In this case, the commutants must be

evaluated by direct computation and it is not possible, in general, to express their structure in a compact form.

We have all the ingredients to evaluate the contribution related to the dissipative generators V_i in (11) in every situation. The case in which all the λ_i vanish but one has been discussed above. The remaining cases can be completely described by considering the following properties.

(i) If $\lambda_i^+ \neq 0$ for several indices i ,

$$\bigcap_{i|\lambda_i^+ \neq 0} \{V_i, V_i^\dagger\}' = \text{span}(\mathbb{I} \otimes \mathbb{I}, \Omega^+). \quad (26)$$

(ii) If $\lambda_i^- \neq 0$ for several indices i ,

$$\bigcap_{i|\lambda_i^- \neq 0} \{V_i, V_i^\dagger\}' = \text{span}(\mathbb{I} \otimes \mathbb{I}). \quad (27)$$

(iii) If $\lambda_i = \lambda_i^+ \neq 0$ and $\lambda_j = \lambda_{j-3}^- \neq 0$ for a pair of indices (i, j) , then

$$\begin{aligned} \{V_i, V_i^\dagger\}' \cap \{V_j, V_j^\dagger\}' = & \quad (28) \\ \begin{cases} \text{span}(\mathbb{I} \otimes \mathbb{I}, \mathbb{I} \otimes \tilde{\sigma}_i, \tilde{\sigma}_i \otimes \mathbb{I}, \tilde{\sigma}_i \otimes \tilde{\sigma}_i), & \text{if } \tilde{\sigma}_i = \tilde{\sigma}_i^\dagger = \hat{\sigma}_j; \\ \text{span}(\mathbb{I} \otimes \mathbb{I}), & \text{otherwise.} \end{cases} \end{aligned}$$

To begin with, we assume $H_S = 0$ and we focus on the dissipative contribution to the dynamics. We denote by n_+ and n_- the number of non-vanishing eigenvalues of the type λ^+ and λ^- respectively. The relevant algebras in the non-trivial cases are summarized in Table I, where the projectors Π are defined as

$$\begin{aligned} \Pi_k &= [\pi_{ij}^k], \quad \pi_{ij}^k = \delta_{ik} \delta_{jk}, \quad k = 1, \dots, 4; \\ \Pi_- &= \frac{1}{4}(\mathbb{I} \otimes \mathbb{I} - \Omega^+), \quad \Pi_+ = \mathbb{I} \otimes \mathbb{I} - \Pi_-. \end{aligned} \quad (29)$$

We notice that $A = B$ is equivalent to $n_- = 0$. We further observe that, in case (iii), $[A, B] = 0$, thus it is

possible to choose $\tilde{U} = \hat{U}$. Moreover, $B = \alpha A$ implies $\tilde{\sigma}_\xi = \hat{\sigma}_\xi$ for the index ξ such that $\lambda_\xi^+ \neq 0$ and $\lambda_\xi^- \neq 0$.

The sets of projectors defined in Theorem 1 are reported in the table. In the remaining cases $\mathcal{M} = \text{span}(\mathbb{I} \otimes \mathbb{I})$, part 1 of Theorem 1 applies and the stationary state is unique. Therefore, the cases described in Table I are necessary conditions for the indirect manipulation of the asymptotic state of the target system T via the auxiliary system P .

III. STATIONARY STATES

We separately explore the non-trivial cases described in Section II. Following Theorem 1 and considering Table I, it is possible to find the family of stationary states ρ_S^∞ , and then to extract the corresponding stationary state of the target subsystem,

$$\rho_T^\infty = \text{Tr}_A \rho_S^\infty, \quad (30)$$

by a partial trace over the degrees of freedom of the auxiliary system. We consider two different choices for the initial state $\rho_S(0)$. In the spirit of the indirect control scheme, it can be a factor state,

$$\rho_S(0) = \rho_T(0) \otimes \rho_A(0), \quad (31)$$

where $\rho_T(0)$ and $\rho_A(0)$ are arbitrary states for the two subsystems, that will be written using a Bloch vector representation as

$$\rho_T(0) = \frac{1}{2} \left(\mathbb{I} + \sum_{k=1}^3 \rho_k^T \sigma_k \right), \quad (32)$$

and analogously for $\rho_A(0)$, with real coefficients ρ_k^T and ρ_k^A . This situation refers to initially uncorrelated systems, that will in general couple during their joint evolution. Alternatively, we consider the pure initial state

$$\rho_S(0) = |\psi\rangle\langle\psi|, \quad |\psi\rangle = \sqrt{P} |\uparrow\rangle \otimes |\uparrow\rangle + \sqrt{1-P} |\downarrow\rangle \otimes |\downarrow\rangle, \quad (33)$$

where $P \in \mathbb{R}$, and $|\uparrow\rangle, |\downarrow\rangle$ are the $+1$, respectively -1 eigenvalues of the operator σ_3 . This state is entangled if $P \neq 0, 1$, and it is maximally entangled if $P = \frac{1}{2}$. It is not an arbitrary entangled state, nevertheless it can show the impact of initial correlations between the two parties on the manipulation of the stationary state of T .

After discussing the four non-trivial cases presented in Table I, we recover the case with $H_S \neq 0$.

A. Case I

In this case, $\lambda_\xi \neq 0$ for some $\xi \in \{1, 2, 3\}$. For the initial state (31), the stationary state of the target system

is given by

$$\begin{aligned} \rho_1^\infty &= u_{\xi 1} \left(\rho_1^T u_{\xi 1} - \rho_2^T u_{\xi 2} + \rho_3^T u_{\xi 3} \right) \\ \rho_2^\infty &= -u_{\xi 2} \left(\rho_1^T u_{\xi 1} - \rho_2^T u_{\xi 2} + \rho_3^T u_{\xi 3} \right) \\ \rho_3^\infty &= u_{\xi 3} \left(\rho_1^T u_{\xi 1} - \rho_2^T u_{\xi 2} + \rho_3^T u_{\xi 3} \right), \end{aligned} \quad (34)$$

where ρ_k^∞ , $k = 1, 2, 3$, are the Bloch vector components of ρ_T^∞ . They depend solely on the initial state of T , therefore, in this case, it is not possible to manipulate ρ_T^∞ by means of an initially uncorrelated auxiliary system. However, if the initial state (33) is taken into account, a dependence is exhibited in terms of the Schmidt coefficient P :

$$\begin{aligned} \rho_1^\infty &= (2P - 1) u_{\xi 1} u_{\xi 3} \\ \rho_2^\infty &= -(2P - 1) u_{\xi 2} u_{\xi 3} \\ \rho_3^\infty &= (2P - 1) u_{\xi 3}^2. \end{aligned} \quad (35)$$

Therefore, if it is possible to vary the initial degree of entanglement, different stationary states for the dynamics result.

B. Case II

In order to fully characterize the stationary states of the dynamics, following Theorem 1 we need to find a stationary state ρ_0 of maximal rank, such that

$$V_\xi \rho_0 V_\xi^\dagger - \frac{1}{2} \{V_\xi^\dagger V_\xi, \rho_0\} = 0, \quad (36)$$

where $\xi \in \{1, 2, 3\}$ satisfies $\lambda_\xi \neq 0$. However, it turns out that every solution of (36) has at least a null eigenvalue. In order to prove this, it is not restrictive to assume that only two coefficients $\tilde{u}_{\xi k}^*$, $k = 1, 2, 3$, are non-vanishing. In fact, the general case reduces to this one by a unitary transformation of the Pauli matrices σ_k . Since the order is not relevant, we assume that

$$\tilde{u}_{\xi 1}^* = e^{i\beta_1} \cos \gamma, \quad \tilde{u}_{\xi 2}^* = e^{i\beta_2} \sin \gamma, \quad \tilde{u}_{\xi 3}^* = 0, \quad (37)$$

with real β_1, β_2 and γ . The general solution of (36) is

$$\begin{aligned} \rho_0 &= \frac{1}{4} \left(\mathbb{I} \otimes \mathbb{I} + r_1 (\mathbb{I} \otimes \sigma_3 + \sigma_3 \otimes \mathbb{I}) + r_2 \sigma_1 \otimes \sigma_1 \right. \\ &\quad \left. + r_3 (\sigma_1 \otimes \sigma_2 + \sigma_2 \otimes \sigma_1) + r_4 \sigma_2 \otimes \sigma_2 + r_5 \sigma_3 \otimes \sigma_3 \right), \end{aligned} \quad (38)$$

where a_i , $i = 1, \dots, 5$ are real, dependent coefficients, such that

$$\begin{aligned} r_1 &= -\frac{1}{2} (4r_5 + 1) \sin(\beta_1 - \beta_2) \sin 2\gamma, \\ r_2 &= \frac{1}{2} \left(4r_5 - 1 - (4r_5 + 1) \cos 2\gamma \right), \\ r_3 &= -\frac{1}{2} (4r_5 + 1) \cos(\beta_1 - \beta_2) \sin 2\gamma, \\ r_4 &= \frac{1}{2} \left(4r_5 - 1 + (4r_5 + 1) \cos 2\gamma \right). \end{aligned} \quad (39)$$

An explicit computation proves that $\text{Det}\rho_0 = 0$ irrespective of r_5 , and then, in full generality, there is not a maximal rank stationary state, in this case. Therefore, it is not possible to apply Theorem 1.

C. Case III

In this case ρ_0 is the solution of

$$V_\xi \rho_0 V_\xi + V_\eta \rho_0 V_\eta - \frac{1}{2} \{V_\xi^2 + V_\eta^2, \rho_0\} = 0, \quad (40)$$

where $\xi \in \{1, 2, 3\}$, $\eta = \xi + 3$ are such that $\lambda_\xi, \lambda_\eta \neq 0$, and $V_\xi^\dagger = V_\xi$, $V_\eta^\dagger = V_\eta$. Since

$$\begin{aligned} V_\xi &= \sqrt{\lambda_\xi} \tilde{\mu}_\xi \mathcal{R}_\xi (\mathbb{I} \otimes \sigma_3 + \sigma_3 \otimes \mathbb{I}) \mathcal{R}_\xi, \\ V_\eta &= \sqrt{\lambda_\eta} \tilde{\mu}_\eta \mathcal{R}_\eta (\mathbb{I} \otimes \sigma_3 - \sigma_3 \otimes \mathbb{I}) \mathcal{R}_\eta, \end{aligned} \quad (41)$$

the general stationary state is found to be

$$\rho_0 = \frac{1}{4} \left(\mathbb{I} \otimes \mathbb{I} + r_1 \mathbb{I} \otimes \tilde{\sigma}_\xi + r_2 \tilde{\sigma}_\xi \otimes \mathbb{I} + r_3 \tilde{\sigma}_\xi \otimes \tilde{\sigma}_\xi \right), \quad (42)$$

where r_i , $i = 1, 2, 3$ are real, independent coefficients, and $r_1 = r_2$ if $\lambda_\xi \neq \lambda_\eta$. The simplest choice leading to a maximal rank stationary state is $r_i = 0$ for all i . From it, following Theorem 1, it is possible to build the complete set of stationary states of the dynamics, and finally to extract the stationary states of T . Although the algebraic structures are different, it turns out that the results are the same of Case I, expressed in (34) and (35).

D. Case IV

In this case, the maximal rank stationary state can be found by solving

$$\sum_k \left(V_k \rho_0 V_k^\dagger - \frac{1}{2} \{V_k^\dagger V_k, \rho_0\} \right) = 0, \quad (43)$$

where the V_k operators correspond to eigenvalues $\lambda_k^+ \neq 0$, and k assumes two or three values in the set $\{1, 2, 3\}$. First of all, we notice that $A \neq A^T$ is a necessary condition for the indirect control of the asymptotic state of T . In fact, in $A = A^T$, all the relevant V_k operators are Hermitian, and then it is possible to choose the maximal rank stationary state as $\rho_0 = \frac{1}{4} \mathbb{I} \otimes \mathbb{I}$, leading to $\rho_k^\infty = 0$ for all k , for both correlated or uncorrelated initial states.

For $A \neq A^T$, the general expression of the maximal rank stationary state ρ_0 is rather involved, therefore we prefer to exhibit a concrete example proving that, in this case, both uncorrelated and correlated initial states allow indirect manipulations of the asymptotic states of T . We thus assume a Kossakowski matrix A of the form

$$A = \begin{bmatrix} a & id & 0 \\ -id & b & 0 \\ 0 & 0 & c \end{bmatrix}, \quad (44)$$

where a, b, c and d are real parameters satisfying the conditions

$$\begin{cases} a \geq 0, & b \geq 0, & c \geq 0, \\ ab - d^2 \geq 0, \end{cases} \quad (45)$$

expressing the complete positivity of the evolution. In this case, the maximal rank stationary state is found to be

$$\begin{aligned} \rho_0 &= \frac{1}{4} \left(\mathbb{I} \otimes \mathbb{I} + r_1 (\mathbb{I} \otimes \sigma_3 + \sigma_3 \otimes \mathbb{I}) \right. \\ &\quad \left. + r_2 (\sigma_1 \otimes \sigma_1 - \sigma_2 \otimes \sigma_2) + r_3 \sigma_3 \otimes \sigma_3 \right), \end{aligned} \quad (46)$$

with

$$\begin{aligned} r_1 &= \frac{2d}{a+b} \\ r_2 &= \frac{(b-a)d^2}{(a+b)(ab+ac+bc)} \\ r_3 &= \frac{(a+b+4c)d^2}{(a+b)(ab+ac+bc)}. \end{aligned} \quad (47)$$

The asymptotic state of T for the uncorrelated initial state has components

$$\begin{aligned} \rho_1^\infty &= 0, & \rho_2^\infty &= 0, \\ \rho_3^\infty &= \frac{r_1}{3+2r_2+r_3} \left(3 + \sum_{k=1}^3 \rho_k^T \rho_k^A \right); \end{aligned} \quad (48)$$

for the correlated initial state we get

$$\begin{aligned} \rho_1^\infty &= 0, & \rho_2^\infty &= 0, \\ \rho_3^\infty &= \frac{4r_1}{3+2r_2+r_3} \left(1 + \sqrt{P(1-P)} \right). \end{aligned} \quad (49)$$

Therefore, in both cases it is possible to manipulate ρ_T^∞ .

Under the assumption $H_S = 0$, the only candidates for the realization of the asymptotic control protocol by means of an auxiliary system are the dynamical systems satisfying the conditions expressed in Case IV, with the additional request $A \neq A^T$. We now discuss the impact of a non-vanishing H_S .

According to (11), when adding the Hamiltonian term, the algebras \mathcal{M} computed in Section II are left unchanged or reduced, depending on the form of H_S . In particular, in Cases II and IV, they are not modified whenever $[H_S, \Omega^+] = 0$, that is H_S is invariant under the exchange $T \leftrightarrow P$ (it contains only terms of the form $\mathbb{I} \otimes \sigma_i + \sigma_i \otimes \mathbb{I}$ or $\sigma_i \otimes \sigma_j + \sigma_j \otimes \sigma_i$, with $i, j = 1, 2, 3$). Otherwise, \mathcal{M} is one-dimensional and part 1 of Theorem 1 applies.

Since in Case III the maximal rank stationary state previously considered is a stationary state without reference of H_S , in Cases I and III the results obtained under a purely dissipative dynamics are valid in general. In Cases II and IV, ρ_0 can be evaluated only after specifying H_S , therefore it is not possible to give general results.

IV. DISCUSSION AND CONCLUSIONS

In this work, we have studied to what extent it is possible to drive the asymptotic states of a system T via an auxiliary system P , following the indirect control approach, when both target and auxiliary systems are two-level systems. We have initially considered the case of a purely dissipative dynamics, and then we have generalized our results in the presence of an Hamiltonian term. We have assumed that the two systems interact separately with a common, homogeneous environment, that is invariant under spatial translation. This choice is expressed by a particular form of the Kossakowski matrix C for the composite system. We have found necessary conditions for the indirect manipulation of the stationary state of T through the initial state of P when the initial state is a product state, and provided a concrete example in which this dependence is indeed apparent. We have also considered the impact of an initial entanglement between the two systems for control purposes.

We have found that this kind of control can be performed only when the blocks of the Kossakowski matrix satisfy the condition $A = B$. This is not merely a mathematical request; in fact, physical system that are well described by a quantum dynamical semigroup in this form can be found in concrete experimental situations, for example in the study of the resonance fluorescence [27, 28], or in the analysis of the weak coupling of two atoms to an external quantum field [29], when the distance between the two atoms can be neglected.

We observe that the phenomenon described in this

work has its origin in the change of the asymptotic behavior of a quantum dynamical semigroup when the system is enlarged. Consider the example presented in the previous section, in Case IV. If only the system T is taken into account, it is described by a dynamical semigroup with the Kossakowski matrix A given in (44). Unless $a = b = d = 0$, this is a uniquely relaxing semigroup with stationary state

$$\rho_1^\infty = 0, \quad \rho_2^\infty = 0, \quad \rho_3^\infty = -\frac{2d}{a+b}. \quad (50)$$

However, when adding the probe system, if $A = B$ and $A \neq A^T$, $S = T + P$ is described by a relaxing semigroup, and the stationary state is not fixed. In these conditions, multiple stationary states for T are generated.

The key ingredient for the controllability in indirect control schemes is given by the correlations between T and P . These correlations can be provided at the beginning, or rather created during the time evolution. We notice that even a purely dissipative evolution can provide the needed correlation (for the asymptotic entanglement in a quantum dynamical semigroup under these hypotheses, see the results presented in [26]). In this sense, in the indirect control approach the environmental action can be considered as a resource, not only a source of noise and decoherence. This kind of behavior has already been observed when dealing with accessibility and controllability of a pair of qubits immersed in a bath of decoupled harmonic oscillators, in an exactly solvable model [22].

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